Chapter 3

Intuitive Calculus

There are two major branches of calculus, differential and integral. The motivation for developing differential and integral calculus comes from solving problems that appear to be unrelated, but whose solutions both employ a common ingredient, the notion of a limit. Differential calculus arises from the problem of understanding rates of change, while integral calculus is first introduced as a tool for finding areas of general regions. The most important and powerful theorem of calculus is called the *fundamental theorem of calculus*, and it is this theorem that provides a surprising and intimate connection between differential and integral calculus.

The purpose of this chapter is to familiarize you with the principle objects of differential and integral calculus. The goal for now is to know what these objects represent graphically. We will provide logical statements that define these objects in later chapters, after which we will be in a position to prove things about them.

Integral Calculus

The process of translating a problem into a mathematical problem is one in which mathematical objects, in particular sets and functions, are introduced to represent the objects of the problem. If you are given the problem of finding the

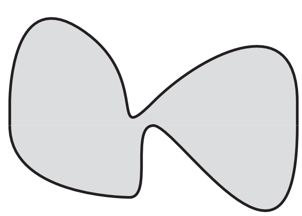

Figure 3.1:

area of the region in Figure 3.1 you might *mathematize* the problem by thinking of the boundary as graphs of functions. The function f is defined for numbers

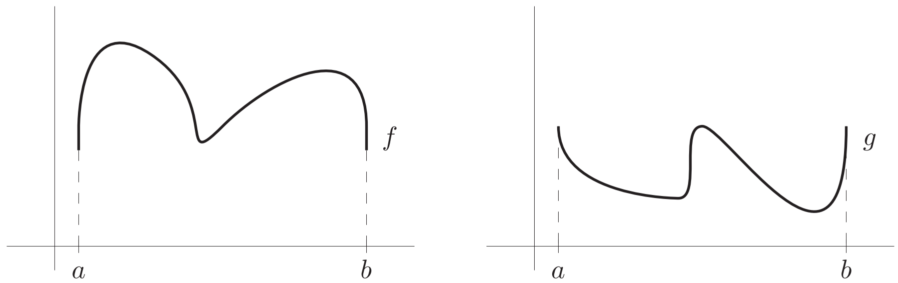

Figure 3.2:

x in the interval $[a, b]$ and its graph is the uppermost boundary of the region. Similarly, g is defined for $x \in [a, b]$ and its graph is the lowermost boundary of the region. The problem has now been put in terms of mathematical objects and it is solved by defining a process, called integration, which produces the area between the graph of a function and the x-axis. The symbol $\int_a^b f$ denotes

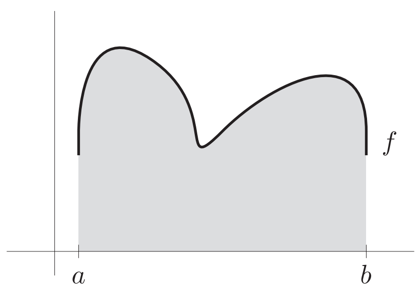

Figure 3.3:

the shaded area in Figure 3.3. You now have a mathematical method to find the area of the region in Figure 3.1, assuming that $\int_a^b f$ and $\int_a^b g$ are computable. The area is then $\int_a^b f - \int_a^b g$ (since $\int_a^b g$ represents the shaded area in Figure 3.4).

The symbol $\int_a^b f$ is called the *integral* of f on the interval $[a, b]$. The integral is itself a function, but it is perhaps unlike functions you have thought about in the past because its domain is not a set of numbers. What one inputs into the integral is a function and an interval of numbers and the integral then outputs a number that represents an area. Actually, we have only given you an idea of what $\int_a^b f$ is for functions whose graph is above the x-axis. When the graph of the function is below the x-axis one defines $\int_a^b f$ to be the *negative* area between the graph and the x-axis. The reason for doing this is to ensure that the integral

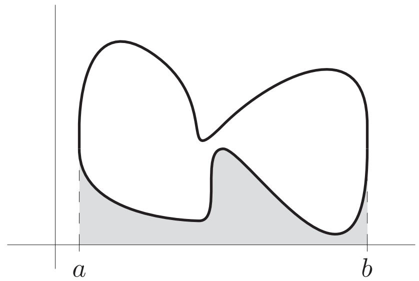

Figure 3.4:

becomes what is called a *linear* function. Linear functions are so important that there is a whole subject of mathematics devoted to them called *linear algebra*. With this extended definition the area of the shaded region in Figure 3.5 is

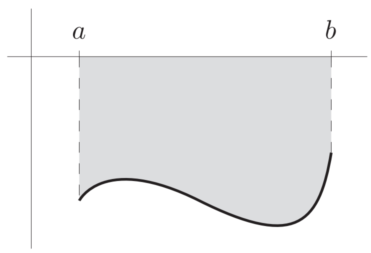

Figure 3.5:

$-\int_a^b f$. The integral is also defined in a way that respects our demand that the area of a whole be the sum of areas of parts. One instance of this is captured in the equation

$$\int_a^c f = \int_a^b f + \int_b^c f, \tag{3.1}$$

for $a < b < c$.

Problem 3.1 *Draw a picture that illustrates the meaning of Equation 3.1 for a function whose graph is above the x-axis.*

Problem 3.2 *Assume f is the function whose graph appears in Figure 3.6. Find the following:* (a) $f(0)$ (b) $\int_0^1 f$ (c) $\int_1^2 f$ (d) $\int_0^2 f$

Differential Calculus

The integral is the principle object of integral calculus and the derivative is the principle object of differential calculus. To acquaint you with the idea of a

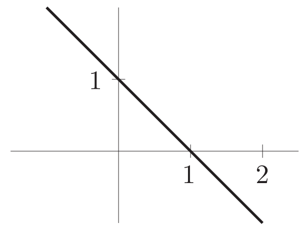

Figure 3.6:

derivative imagine that a car is driving along a straight road. A mathematization of this scenario might be to think of the road as a number line, with the origin at some specified place (see Figure 3.7). One might then let f represent the

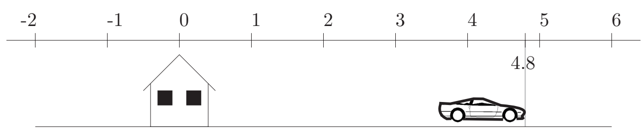

Figure 3.7: Snapshot at time 3

function that outputs the number on the line where the car is when the time is input. Perhaps at time 3 (units are not particularly relevant) the car is at 4.8, which can be expressed briefly by writing $f(3) = 4.8$. Of course there are other functions around; we could let g represent the function that returns the car's velocity when the time is input. The idea of the derivative is that g can be obtained from f. In fact, if you look at the graph of f it is possible to read off (at least approximately) what $g(t)$ is at any time t. The illustration in Figure 3.8 shows you how this can be done. Here is why this works; if you

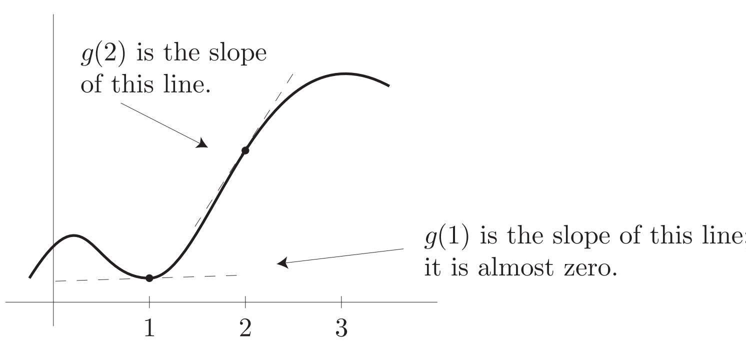

Figure 3.8: The graph of f

were asked to find the average velocity of the car in the interval of time between $t = 1$ and $t = 3$ you would divide the change of position by the elapsed time. In terms of our symbols this is exactly

$$\frac{f(3) - f(1)}{3 - 1}.$$

In terms of the graph of f this number is the slope of the secant line joining two points on the graph (see Figure 3.9). If you want to know the velocity at the

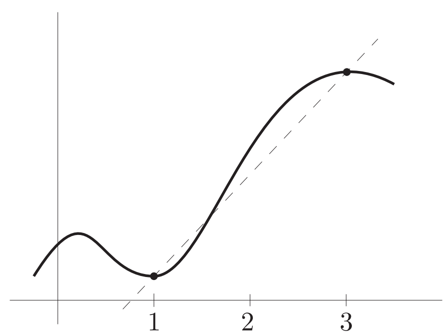

Figure 3.9: The slope of the line represents an average velocity.

instant when $t = 1$ you could approximate it by taking averages over shorter and shorter time intervals (Figure 3.10). The slopes of the secant lines then

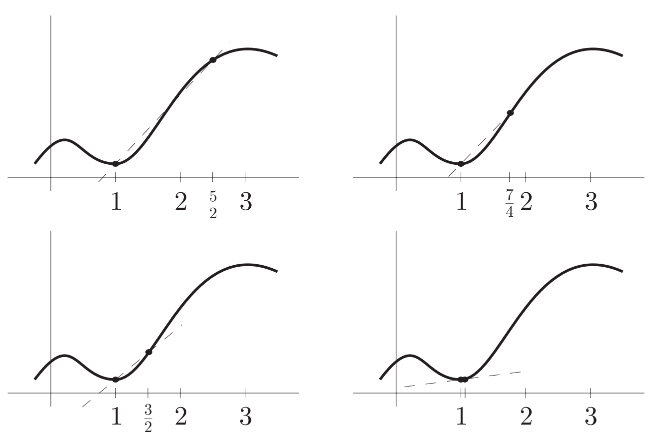

Figure 3.10: Instantaneous velocity is a limit of average velocities.

approach the slope of the line tangent to the graph of f at $(1, f(1))$, and the slope of this tangent line is $g(1)$. If you are given a function f, the *derivative* of

f is denoted f' and it is another function. When x is input into the derivative the output is the slope of the tangent line to the graph of f at the point $(x, f(x))$.

In the preceding discussion the function g was the derivative of f, which may be expressed symbolically by writing $g = f'$. The word "tangent" can be misleading in certain instances, such as when the graph of the function is a straight line. A better interpretation of what the derivative function outputs is obtained as follows; if you are interested in $f'(3)$, look at the graph of f around the point $(3, f(3))$ and zoom in. If upon repeated magnification the graph approaches a line, as in Figure 3.11, then $f'(3)$ is the slope of that line. A particular instance is when the graph of f is itself a line, in which case $f'(3)$ is the slope of that line. For example, the function that is defined by the equation $f(x) = 5x$ has a derivative function that outputs the number 5, no matter what number is input. If upon magnification the graph does not approach a line (see Figure 3.11 again), then we say that $f'(3)$ does not exist. This viewpoint of a derivative is a very good one; if $f'(x)$ exists, we interpret this by thinking of f as behaving like a linear function (a function whose graph is a line) for numbers close to x.

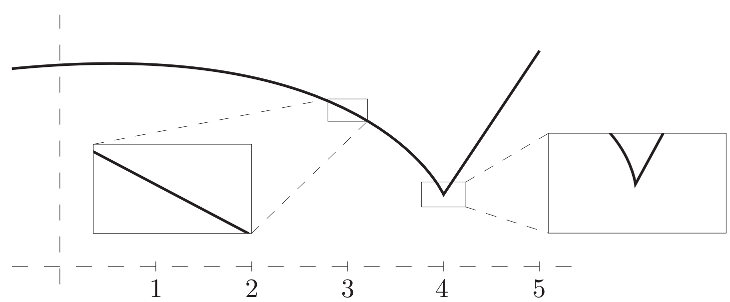

Figure 3.11: The derivative at 3 exists, but the derivative does not exist at 4

Problem 3.3 *If f is the function whose graph appears in Figure 3.12 then find* *(a)* $f(2)$ *(b)* $\int_0^1 f$ *(c)* $f'(3)$ *(d)* $f'(1)$ *(e)* $\frac{f(3)-f(1)}{3-1}$ *(f)* $\frac{f(4)-f(3)}{4-3}$

The Fundamental Theorem

The two branches of calculus described above appear to have nothing to do with each other, but actually they are intimately related. The relation between the integral and differential calculus is the content of the *fundamental theorem of calculus*, which provides a careful mathematical formulation of the relationship in the *statement* of the theorem, together with the formal mathematical *proof* that the relationship is true. We have not yet defined the concepts needed to give a formal statement of the theorem, but we are in a position to provide an

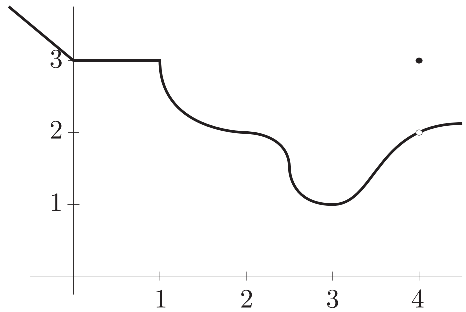

Figure 3.12:

intuitive description of the statement. The goal of the next several chapters is to build the mathematical tools needed to give a precise statement and proof of the fundamental theorem.

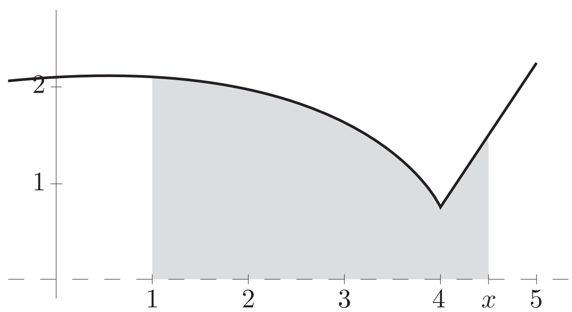

Figure 3.13: If $g(x)$ is the area of the shaded region above, then the graph must be the one for g's derivative.

If you have a function f like the one pictured in Figure 3.11, then it is possible to use the theory of integration to define a related function g that outputs the area bounded by the graph of f on the interval $[1, x]$, as illustrated in Figure 3.13. Notice that g is a function of the interval of integration; imagine x changing, so that the interval of numbers $[1, x]$ changes, and visualize the corresponding shaded area changing. When $x = 1$ there is absolutely no area, so $g(1) = 0$. As x moves to the right, $g(x)$ gets larger. The fundamental theorem states that when one associates such a function g to a function f in this manner, it frequently happens that the derivative of g is f. We use the word "frequently" because there are functions f where this constructions yields a function g with $g' = f$, and then there are functions f where this construction fails altogether. The fundamental theorem gives a precise condition on f that guarantees the resulting function g satisfies $g' = f$.

The practical importance of the fundamental theorem is that it provides a means by which it becomes possible to *calculate* integrals, and this is where the name of the subject *calculus* derives from. If we desperately needed to know what $\int_1^5 f$ is, then the definition of g tells us that the answer is $g(5)$, and the fundamental theorem tells us that g has f as its derivative. It turns out that if we find any function h that has f as its derivative, this function h differs from g by a constant, that is $g(x) - h(x)$ is the same value no matter what x is input (this is a fact that will be established when we develop differential calculus). If we put this information together it gives a recipe for finding $\int_1^5 f$. The first step is to find any function h that satisfies $h' = f$. The fact that $g(5) - h(5) = g(1) - h(1)$ then leads immediately to the answer

$$h(5) - h(1) = g(5) - g(1) = g(5) = \int_1^5 f.$$

This reduces the problem of finding areas to the computational task of finding *antiderivatives*.

Problem 3.4 *Assume f is the function whose graph is illustrated in Figure 3.13 and g is the corresponding function defined by*

$$g(x) = \int_1^x f.$$

Find approximate values for $f(2)$, $g(2)$, $g'(3)$, and $g'(4)$.

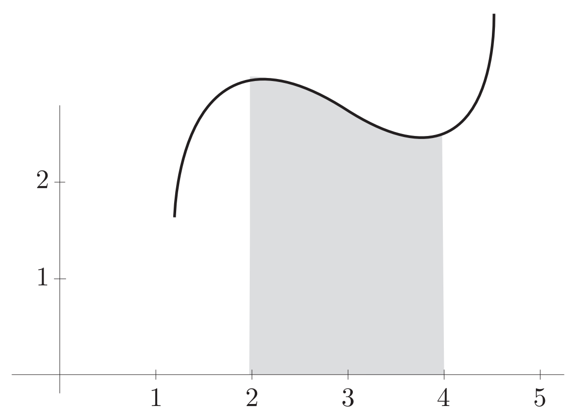

Figure 3.14: The graph of f.

Problem 3.5 *Assume that f is the derivative of the function g that returns an output of $\frac{x^4}{4} - 3x^3 + 13x^2 - 21x$ when x is input, i.e. $g(x) = \frac{x^4}{4} - 3x^3 + 13x^2 - 21x$ for every x. The graph of f is pictured in Figure 3.14; find the area of the region shaded in Figure 3.14.*